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Convergence of Some General Iterative Schemes

Bhagwati Prasad and Ritu Sahni

Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida-201307, India
b_prasad10@yahoo.com; bhagwati.prasad@jiit.ac.in

Abstract

Many problems in engineering and applied sciences can be formulated as fixed-point problems. For example, any nonlinear equation $f(x) = 0$ can be rearranged as a fixed-point equation in the form of $g(x) = f(x) + x = x$. The solution to such type of equation is computed iteratively through some iterative procedure. In this paper we discuss the Ishikawa and improved Ishikawa iterative schemes for solving the nonlinear equations motivated by the results of Biazar and Amriteimoori [3].

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1. Introduction

Let X be a metric space and $T : X \to X$ and $x_0 \in X$. Assume that $x_{n+1} = Tx_n$, $n = 0,1,2,...$ involving T, which yields a sequence $\{x_n\}$ in X. This is one of the most popular iterative procedures used in the literature usually called the function iteration or Picard iteration. When the contractive conditions are slightly weaker, the Picard iteration need not converge to a fixed point of the operator under consideration and some other iterative procedures should be slightly weaker, the Picard iteration need not converge to a fixed point of the operator under consideration and some other iterative procedures should be considered. This results in the generalization of the function iteration in various ways in different settings. The Mann iteration scheme [6], for $n = 0,1,2,...$ and $\alpha_n \subset [0,1],$ is defined as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad (1)$$

This is further generalized by Ishikawa [4] in the following way:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2}$$

where α_n, $\beta_n \subset [0,1]$ and $n = 0,1,2,\ldots$.

Further, these iterative schemes are developed by taking two mappings $S, T : Y \to X$, where $T(Y) \subseteq S(Y)$ and $x_0 \in Y$. Singh et al [11] discuss the following iterative procedure:

$$Sx_{n+1} = f(T, x_n),\ n = 0,1,\ldots \tag{3}$$

It is called Jungck-iterative procedures (also see [5]). If $f(T, x_n)$ in (3) is replaced by Tx_n and $(1-\alpha_n)Sx_n + \alpha_n Tx_n$, it becomes Jungck-Picard and Jungck-Mann iteration respectively.

Jungck–Ishikawa iteration (see [7]) is defined in the following manner:

$$\left.\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n T z_n \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n T x_n \end{aligned}\right\} \tag{4}$$

For a detailed discussion and stability of various iterative schemes, see Berinde [2], Rhoades [8], Rhoades and Saliga [9], Singh and Prasad [10] and Singh et al. [11] and references thereof.

2. Preliminaries

Following Biazar and Amriteimoori [3], we define the basic concepts and relevant results required in the sequel. In [3], the fixed point iteration for Picard iteration is shown to be improved under following conditions.

(i) Initial approximation is chosen in the interval $[a,b]$, where function is defined.

(ii) Function has continuous derivative on (a,b).

(iii) $|T'(x)| < 1$ for all $x \in [a,b]$

(iv) $a \le T(x) \le b$ for all $x \in [a,b]$.

Definition 2.1 [3]. Let $\{x_n\}$ converge to α . If there exist an integer constant *p*, and real positive constant *C* such that

$$\lim_{n\to\infty}\left|\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}\right| = C.$$

Then p is called the order and C the constant of convergence.

Theorem 2.1 ([1], [3]). Suppose $T \in C^p[a,b]$. If $T^{(k)}(x)=0$, for $k = 1, 2, ..., p-1$ and $T_p(x) \neq 0$, then the sequence $\{x_n\}$ is of order p.

Biazar and Amriteimoori [3] approximated the root $\alpha \in (a,b)$ by taking $\alpha \cong \frac{a+b}{2}$.

To improve the order of convergence of the fixed point iterative method, such that $T'(\alpha)=T''(\alpha)=...=T^k(\alpha)=0$, determine $\lambda_i\,(i=1,2,...,k)$ from the following equation

$$x+\lambda_1 x+\lambda_2 x^2+...\lambda_k x^k = T(x)+\lambda_1 x+\lambda_2 x^2+...\lambda_k x^k .$$

It can be written in the form of a fixed point equation as

$$x=\frac{T(x)+\lambda_1 x+\lambda_2 x^2+...\lambda_k x^k}{1+\lambda_1+\lambda_2 x+...+\lambda_k x^{k-1}}=T_\lambda(x) \cdot$$

The assumption $T_\lambda'(\alpha)=T_\lambda''(\alpha)=...=T_\lambda{}^k(\alpha)=0$, yields to a system of linear equations, which after solving (see [3]) reduced to an upper triangular matrix with nonzero diagonal entries. Therefore, its determinant is non-zero. Hence $\lambda_i\,(i=1,2,...k)$ can be determined uniquely.

Now we define the improved iteration schemes such as improved Picard in the following manner

$$x_{n+1}=T_\lambda x_n,\ n=0,1,2,...$$

where, $$T_\lambda(x)=\frac{T(x)+\lambda_1 x+\lambda_2 x^2+...\lambda_k x^k}{1+\lambda_1+\lambda_2 x+...+\lambda_k x^{k-1}} \tag{5}$$

The improved Ishikawa iteration is defined as

$$\begin{aligned} x_{n+1}&=(1-\alpha_n)x_n+\alpha_n T_\lambda y_n \\ y_n&=(1-\beta_n)x_n+\beta_n T_\lambda x_n, \quad n=0,1,2,... \end{aligned} \tag{6}$$

For Jungck iterative schemes, given $S,T:Y\to X$, $T(Y)\subseteq S(Y)$ and $x_0 \in Y$, some of the improved Jungck iterative schemes are defined as follows:

(i) $f(T,x_n)=T_\lambda x_n$, called improved Jungck-Picard iteration.

(ii) $f(T,x_n)=(1-\alpha_n)Sx_n+\alpha_n T_\lambda x_n$, called improved Jungck-Mann iteration.

(iii)

$$\left.\begin{aligned} f(T,\ x_n)&=(1-\alpha_n)Sx_n+\alpha_n T_\lambda z_n \\ Sz_n&=(1-\beta_n)Sx_n+\beta_n T_\lambda x_n \end{aligned}\right\} \tag{7}$$

called improved Jungck –Ishikawa iteration.

3. Experiments

Consider the equation $f(x) = e^{(1-x)^2} - 1 - x = 0$, which has a unique root in the interval (0, 1). Take $\alpha \cong 0.5$ and convert the equation into two parts such that $Sx = x$ and $Tx = e^{(1-x)^2} - 1$. We apply Ishikawa (2), Jungck-Ishikawa (4) and improved Jungck-Ishikawa iteration (7) for solving it. Further, we solve it by taking

$$T_\lambda(x) = \frac{e^{(1-x)^2} - 1 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 can be determined from the following system of linear equations:

$$\begin{bmatrix} 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\ 0 & 1 & 1 & 0.75 & 0.5 \\ 0 & 0 & 2 & 3 & 1.5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} -T'(\alpha) \\ -T^{(2)}(\alpha) \\ -T^{(3)}(\alpha) \\ -T^{(4)}(\alpha) \\ -T^{(5)}(\alpha) \end{bmatrix} = \begin{bmatrix} 1.28403 \\ -3.85208 \\ 8.98818 \\ -32.1006 \\ 104.006 \end{bmatrix}$$

Thus we have $\lambda_1 = 6.0858$, $\lambda_2 = -17.7757$, $\lambda_3 = 22.2698$, $\lambda_4 = -14.0173$ and $\lambda_5 = 4.3336$. Results using these iterative procedures are given in the following tables:

TABLE I
CONVERGENCE OF ISHIKAWA AND IMPROVED ISHIKAWA ITERATION

	Ishikawa iteration		**Improved Ishikawa iteration**	
n	x_{n+1}	Tx_n	x_{n+1}	$T_\lambda x_n$
0	0.452503	0.284025	0.419943	0.405441
1	0.433679	0.349525	0.413164	0.4124
2	0.42431	0.378119	0.412472	0.412398
	.	.	.	.
6	.	.	0.412391	0.412391
.	.	.	.	.
.	.	.	.	.
24	0.412391	0.412391	.	.

TABLE II
CONVERGENCE OF JUNGCK-ISHIKAWA ITERATION

Jungck-Ishikawa iteration			
n	x_{n+1}	Sx_n	Tx_{n+1}
0	0.333256	0.284025	0.333256
1	0.498394	0.559784	0.498394
2	0.334561	0.286093	0.334561
.	.	.	.
.	.	.	.
100	0.366681	0.337478	0.366681

TABLE III
CONVERGENCE OF IMPROVED JUNGCK-ISHIKAWA ITERATION

Improved Jungck-Ishikawa iteration			
n	x_{n+1}	Sx_n	$T_\lambda x_{n+1}$
0	0.407871	0.405441	0.407871
1	0.412678	0.412327	0.412678
2	0.412373	0.412394	0.412373
.	.	.	.
.	.	.	.
5	0.412391	0.412391	0.412391

4. Conclusions

The improved Ishikawa iteration scheme converges just in six steps whereas it takes twenty four steps in the usual Ishikawa scheme. The Jungck-Ishikawa does not converge (see Table II) in this case while the improved Jungck-Ishikawa converges only in five steps. Thus the improved Ishikawa and the improved Jungck-Ishikawa schemes seem to have almost same rate of convergence in this case.

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